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Quantum Axiomatics and a Theorem of M. P. Solèr†

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Three of the traditional quantum axioms (orthocomplementation, orthomodularity, and the covering law) show incompatibilities with two products introduced by Aerts for the description of joint entities. Inspired by Solèr's theorem and Holland's AUG axiom, we propose a property of 'plane transitivity,' which also characterizes classical Hilbert spaces among infinite-dimensional orthomodular spaces, as a possible partial substitute for the 'defective' axioms.

1. INTRODUCTION

In his axiomatization of standard quantum mechanics Holland (1995) introduced the Ample Unitary Group Axiom [cf. condition (2) in Proposition 1 of the present paper]. It hints at an evolution axiom, but has the shortcoming that it is not lattice-theoretic. In particular, it cannot be formulated for *property lattices*—complete, atomistic, and orthocomplemented lattices—which play a central role in the Geneva–Brussels approach to the foundations of physics (Piron, 1976, 1989, 1990; Aerts, 1982, 1983, 1984; Moore, 1995, 1999). Inspired by this axiom, we propose a property, called 'plane transitivity' (Section 4), which does not have this imperfection. Like the AUG axiom, it characterizes classical Hilbert spaces among infinite-dimensional orthomodular spaces and it still seems to demand 'enough symmetries or evolutions.'

Traditional quantum axiomatics show some shortcomings in the description of compound systems (Aerts, 1982, 1984; Pulmannovà, 1983, 1985). In particular, orthocomplementation, orthomodularity, and the covering law are not compatible with two products—'separated' and 'minimal'—introduced by Aerts (Section 3). Plane transitivity, on the other hand, 'survives' these

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two products (Section 4), and is a candidate to help fill the gap left by the failing axioms.

2. ALTERNATIVES TO SOLÈR'S THEOREM

Consider a complete, atomistic, orthocomplemented, and irreducible lattice $\mathscr{L}$ satisfying the covering law. Suppose, moreover, that its length is at least 4. Then there exist a division ring K with an involutorial antiautomorphism $\lambda \mapsto \lambda^*$ and a vector space E over K with a Hermitian form $\langle \cdot, \cdot \rangle$ such that $\mathscr{L}$ is orthoisomorphic to the lattice $\mathscr{L}(E)$ of closed (biorthogonal) subspaces of E. Moreover, $\mathscr{L}$ is orthomodular if and only if $(E, K, \langle \cdot, \cdot \rangle)$ is orthomodular: $M + M^{\perp} = E$ for every $\emptyset \neq M \subset E$ with $M = M^{\perp\perp}$ (Maeda and Maeda, 1970; Piron, 1976; Faure and Frölicher, 1995). Solèr (1995) has proven the following characterization of classical Hilbert spaces: If E contains an infinite orthonormal sequence, then $K = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and $(E, K, \langle \cdot, \cdot \rangle)$ is the corresponding Hilbert space. Holland (1995) has shown that it is enough to demand the existence of a nonzero $\lambda \in K$ and an infinite orthogonal sequence $(e_n)_n \in E$ such that $\langle e_n, e_n \rangle = \lambda$ for every n. To be precise, either $(E, K, \langle \cdot, \cdot \rangle)$ or $(E, K, -\langle \cdot, \cdot \rangle)$ is then a classical Hilbert space. We shall not make this precision explicitly in what follows.

In the following proposition, we summarize some alternatives to Solèr's result, by means of automorphisms of $\mathscr{L}(E)$.

Proposition 1. Let $(E, K, \langle \cdot, \cdot \rangle)$ be an orthomodular space and let $\mathscr{L}(E)$ be the lattice of its closed subspaces. The following are equivalent:

(1) $(E, K, \langle \cdot, \cdot \rangle)$ is an infinite-dimensional Hilbert space over $K = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

(2) E is infinite-dimensional, and given two orthogonal atoms p, q in $\mathscr{L}(E)$, there is a unitary operator U such that $U(p) = q$.

(3) There exist $a, b \in \mathscr{L}(E)$, where b is of dimension at least 2, and an ortholattice automorphism f of $\mathscr{L}(E)$ such that $f(a) \leq a$ and $f|_{[0,b]}$ is the identical map.

(4) E is infinite-dimensional, and given two orthogonal atoms p, q in $\mathscr{L}(E)$, there exist distinct atoms $p_1 p_2$ and an ortholattice automorphism f of $\mathscr{L}(E)$ such that $f|_{[0, p_1 \vee p_2]}$ is the identity and $f(p) = q$.

Condition (2) is Holland's (1995) Ample Unitary Group axiom and (3) is due to Mayet (1998). Using the properties listed in Section 2 of Mayet (1998), one can easily prove that (4) implies (2). We will use (4) to formulate a lattice-theoretic alternative to the AUG axiom (Section 4).

3. COMPOUND ENTITIES AND THE AXIOMS

Aerts has introduced two 'products' for the description of compound entities. We shall present them 'mathematically' and recall their 'interaction' with the axioms of quantum mechanics proposed by Piron (1976). For an operational justification of these products we refer to Aerts (1982, 1984).

First we recall some notions and results due to Moore (1995). A *state space* is a pair $(\Sigma, \perp)$, where Σ is a set (of states) and $\perp$ (orthogonality) is a symmetric, antireflexive, binary relation which separates the points of Σ (if $p \neq q$, then $\exists r$ such that $p \perp r$ and $q \not\perp r$). For $A \subset \Sigma$, put $A^{\perp} = \{q \in \Sigma | \; q \perp p \; \forall p \in A\}$. Then $(\mathcal{L}_{\Sigma}, \subset, {}^{\perp})$ is a property lattice with $\bigwedge\{A_r\} = \bigcap\{A_r\}$, where $\mathcal{L}_{\Sigma} = \{A \subset \Sigma | A = A^{\perp\perp}\}$. In particular, $(\Sigma, \mathcal{L}_{\Sigma})$ is a T_1*-closure* space: $\mathcal{L}_{\Sigma} \ni \emptyset$ is a family of subsets of Σ closed under arbitrary intersections, and $\{p\} \in \mathcal{L}_{\Sigma}, \forall p \in \Sigma$.

Next, consider two entities S_1, S_2 described by their state spaces $(\Sigma_1, \perp_1)$ and $(\Sigma_2, \perp_2)$. Denote the corresponding property lattices by $\mathcal{L}_1$ and $\mathcal{L}_2$. Suppose S_1 and S_2 are 'separated.' Aerts (1982) suggested the *separated product* $\mathcal{L}_1 \oslash \mathcal{L}_2$ for the description of S_1 and S_2 taken together. Its state space is $(\Sigma_1 \times \Sigma_2, \perp_{\oslash})$, where

$$(p_1, p_2) \perp_{\oslash} (q_1, q_2) \Leftrightarrow p_1 \perp q_1 \text{ or } p_2 \perp q_2$$

$\mathcal{L}_1 \oslash \mathcal{L}_2$ is then the corresponding property lattice (Piron, 1989). This product is not 'compatible' with orthomodularity and the covering law in the following sense: If $\mathcal{L}_1 \oslash \mathcal{L}_2$ satisfies one of these properties, then $\mathcal{L}_1$ or $\mathcal{L}_2$ is Boolean (Aerts, 1982).

Aerts (1984) proposed another lattice as the 'coarsest' description of a compound entity containing the two (not necessarily separated) entities S_1, S_2. We give a slightly different, but equivalent construction of this *minimal product* $\mathcal{L}_1 \amalg \mathcal{L}_2$. Consider the closure spaces $(\Sigma_i, \mathcal{L}_i)$. Since $\mathbf{Cls}_1$, the category of T_1-closure spaces and continuous maps, is closed under products (Dikranjan *et al.*, 1988), $(\Sigma_1, \mathcal{L}_1)$ and $(\Sigma_2, \mathcal{L}_2)$ have a $\mathbf{Cls}_1$-product, which we denote $(\Sigma_1 \times \Sigma_2, \mathcal{L}_1 \amalg \mathcal{L}_2)$. This notation is for consistency with (Aerts *et al.*, 1999). Of course, $\mathcal{L}_1 \amalg \mathcal{L}_2$ is a complete atomistic lattice, but the orthocomplementation is problematic. Indeed, if we define the following—operationally justified by Aerts (1984)—orthogonality on $\Sigma_1 \times \Sigma_2$: $(p_1, p_2) \perp (q_1, q_2) \Leftrightarrow p_1 \perp q_1$ or $p_2 \perp q_2$, then $\mathcal{L}_1 \amalg \mathcal{L}_2$ cannot have an orthocomplementation compatible with $\perp$ unless $\mathcal{L}_1 \subset \{0, 1\}$ or $\mathcal{L}_2 \subset \{0, 1\}$. Moreover, the same is true for the covering law: If $\mathcal{L}_1 \amalg \mathcal{L}_2$ satisfies the covering law, then $\mathcal{L}_1$ or $\mathcal{L}_2$ is trivial. For completeness, we mention that this product is compatible with a suitable form of orthomodularity.

These problems with the traditional axioms in the description of joint entities have made it desirable to find (nice) properties compatible with the separated and minimal products. If we slightly generalize condition (4) of Proposition 1, we obtain a property which survives both products.

4. PLANE TRANSITIVITY

To encompass both products with the same terminology, we introduce *pseudo property lattices* (ppl's). A lattice ($\perp$ Σ, $\mathscr{L}$,) is a ppl if $\mathscr{L}$ is a complete atomistic lattice and $\perp$ is an orthogonality on its set of atoms Σ. Using the well-known correspondence between atomistic lattices and T_1-closure spaces (Faure, 1994), we have that every ppl has an associated closure space (Σ, $\mathscr{F}_{\mathscr{L}}$), where

$$\mathscr{F}_{\mathscr{L}} = \{F \subset \sum | \; p \in \sum, p < \vee F \Rightarrow p \in F\}$$

It easily follows that the above construction of the minimal product generalizes to a minimal product of ppl's. To be precise, the minimal product of ($\mathscr{L}_1$, Σ_1, $\perp$) and ($\mathscr{L}_2$, Σ_2, $\perp$) is ($\mathscr{L}_1 \amalg \mathscr{L}_2$, $\Sigma_1 \times \Sigma_2$, $\perp$), where ($\Sigma_1 \times \Sigma_2$, $\mathscr{L}_1 \amalg \mathscr{L}_2$) is the $\mathbf{Cls}_1$-product of (Σ_1, $\mathscr{F}_{\mathscr{L}_1}$) and ($\Sigma_2$, $\mathscr{F}_{\mathscr{L}_2}$), and the orthogonality is defined as above.

We call $f: \mathscr{L} \to \mathscr{L}$ a *symmetry* (of ppl's) if it is an order-automorphism, such that $\forall p, q \in \Sigma$ we have $p \perp q \Leftrightarrow f(p) \perp f(q)$. We remark that for state spaces, symmetries are nothing other than permutations conserving the orthogonality in both directions (Piron, 1989). Indeed, if α is such a permutation of (Σ, $\perp$), then

$$f: \quad \mathscr{L}_\Sigma \to \mathscr{L}_\Sigma: \quad A \mapsto \alpha(A)$$

is the unique orthoautomorphism of $\mathscr{L}_\Sigma$ such that $f\{p\} = \alpha(p)$ for every p in Σ. In particular, f is a symmetry of the ppl ($\mathscr{L}_\Sigma$, Σ, $\perp$) associated to (Σ, $\perp$).

We call a ppl ($\mathscr{L}$, Σ, $\perp$) *plane transitive* if for all atoms p, $q \in \Sigma$ there exist two distinct atoms p_1, p_2 and a symmetry f such that $f|_{[0,p_1 \vee p_2]}$ is the identity and $f(p) = q$. Looking at Proposition 1, it is obvious that if $\mathscr{L}$ is the lattice of biorthogonal subspaces of an infinite-dimensional orthomodular space E, E is a classical Hilbert space iff (with a slight abuse of language) $\mathscr{L}$ is plane transitive.

Proposition 2. Let ($\mathscr{L}_1$, Σ_1, $\perp$) and ($\mathscr{L}_2$, Σ_2, $\perp$) be ppl's. If both are plane transitive, then so is their minimal product ($\mathscr{L}_1 \amalg \mathscr{L}_2$, $\Sigma_1 \times \Sigma_2$, $\perp$).

Indeed, consider (r_1, r_2) and (s_1, s_2) in $\Sigma_1 \times \Sigma_2$. Choose a symmetry f_1 and an atom $p_1 \in \Sigma_1$ such that $f_1(r_1) = s_1$ and $f_1(p_1) = p_1$. Next, choose $p_2 \neq q_2$ in Σ_2 and a symmetry f_2 of ($\mathscr{L}_2$, Σ_2, $\perp$) such that $f_2(r_2) = s_2$ and $f|_{[0,p_2 \vee p_2]}$ is the identical map. Then $f_1|_{\Sigma_i}$ is a $\mathbf{Cls}_1$-automorphism of (Σ_i, $\mathscr{F}_{\mathscr{L}_i}$). It follows that $(t_1, t_2) \mapsto (f_1(t_1), f_2(t_2))$ is a $\mathbf{Cls}_1$-automorphism of ($\Sigma_1 \times$

Σ_2, $\mathscr{L}_1 \amalg \mathscr{L}_2$) and hence generates an order-automorphism $f_1 \times f_2$ of $\mathscr{L}_1 \amalg \mathscr{L}_2$. Trivially, $f_1 \times f_2(r_1, r_2) = (s_1, s_2)$. Also, $f_1 \times f_2|_{[0,(p_1,p_2)\vee(p_1,q_2)]}$ is the identity. Finally, it is straightforward to verify that $f_1 \times f_2$ conserves the orthogonality on $\Sigma_1 \times \Sigma_2$ in both directions.

Using a similar argument, one easily shows the same holds for the separated product. Note that a state space (Σ, $\perp$) is called plane transitive if its associated ppl ($\mathscr{L}_\Sigma$, Σ, $\perp$) is plane transitive.

Proposition 3. If two state spaces (Σ_1, $\perp$) and (Σ_2, $\perp$) are plane transitive, then so is their separated product ($\Sigma_1 \times \Sigma_2$, $\perp_\otimes$).

5. QUESTIONS

Several questions remain. Plane transitivity does not have the necessary elegance to be a fundamental axiom: What is the physical significance of this invariant plane? Another question is: Can the unitary operators of an orthomodular space be characterized at the lattice level? In other words, can Holland's AUG axiom be formulated lattice-theoretically? Perhaps it can be generalized to the transitivity of the whole group of ortholattice automorphisms and still characterize classical Hilbert spaces among infinite-dimensional orthomodular spaces. This would be an elegant symmetry (or evolution) axiom.

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